A general Euclidean connection for the SO(n,m) Lie algebra and the algebraic approach to scattering

1995 J. Phys. A: Math. Gen. 28 3743

(http://iopscience.iop.org/0305-4470/28/13/019)

J. Phys. A: Math. Gen. 28 (1995) 3743–3747. Printed in the UK

A general Euclidean connection for the so(n, m) Lie algebra and the algebraic approach to scattering

C Hategan and R A Ionescu
Institute of Atomic Physics, Bucharest, Romania

Received 13 September 1994, in final form 30 January 1995

Abstract. We obtain a general Euclidean connection for the so(n, m) Lie algebra. This Euclidean connection allows an algebraic derivation of the S-matrix and it reduces to the known connection in suitable circumstances.

1. Introduction

The Euclidean connection has played an important role in the algebraic approach to scattering [1]. In this framework the S-matrix of a scattering system may be evaluated by establishing a connection between the operators of the potential algebra describing the interaction potential and the operators of an appropriate asymptotic algebra describing the long-range behaviour of the system. This connection formula has been referred to as the Euclidean connection. The algebraic approach was applied to heavy ion collisions [2], nuclear reactions [3] and to relativistic systems [4]. In these algebraic approaches the potential algebra operators are expressed as second-order polynomials in the asymptotic algebra operators [1] and it leads to an S-matrix which is a ratio of Γ functions. Although this is the case for all of the problems which can be treated algebraically (the Pöschl–Teller potential, the Morse potential, the Coulomb potential), we shall prove that this is not the general case. The fact that the usual Euclidean connection works for all known examples is due to the expression of the potential algebra operators as second-order differential operators.

The purpose of this paper is to generalize the usual Euclidean connection and to extract an S-matrix for the scattering with an so(n, m) potential algebra. Also, we shall obtain geometric Hamiltonians which are related to this Euclidean connection.

2. General Euclidean connections

2.1. *Euclidean connection for the so(n, m) Lie algebra*

We consider the potential Lie algebra so(n, m) and we denote by L_i the operators of the so(n) Lie subalgebra and by V_α the operators of the so(m) Lie subalgebra. Also, we take $e(n) \oplus e(m)$ to be the asymptotic Lie algebra where $e(n)$ is the Euclidean Lie algebra in n dimensions with generators l_i and p_i, and $e(m)$ is the Euclidean Lie algebra in m dimensions with generators v_α and π_α. It is a straightforward but tedious exercise to prove that the operators

$$
\begin{aligned}
L_i &= l_i \\
V_\alpha &= v_\alpha \\
A_{\mathrm{i}\alpha} &= \frac{1}{k}\mathrm{e}^{\mathrm{i}\phi(k,C_{\mathrm{so}(n)}+C_{\mathrm{so}(m)})}\left\{\frac{1}{2\mathrm{i}}[C_{\mathrm{so}(n)}+C_{\mathrm{so}(m)},\, p_i\pi_\alpha]+\eta(k)p_i\pi_\alpha\right\}\mathrm{e}^{-\mathrm{i}\phi(k,C_{\mathrm{so}(n)}+C_{\mathrm{so}(m)})}
\end{aligned}
\tag{1}
$$

are a Hermitian representation of the so(n, m) algebra with

$$C_{\mathrm{so}(n,m)} = L^2 + V^2 - \sum_{\mathrm{i}\alpha} A_{\mathrm{i}\alpha}^2 = \omega(\omega + n + m - 2)$$

where $\omega = -(n+m-2)/2 + \mathrm{i}\eta$ provided that l_i, p_i give a Hermitian representation of the $e(n)$ algebra with $p_i p^i = k^2$, and v_α, π_α give a Hermitian representation of the $e(m)$ algebra with $\pi_\alpha \pi^\alpha = 1$. If $\phi(k, C_{\mathrm{so}(n)} + C_{\mathrm{so}(m)}) \equiv 0$ the above Euclidean connection reduces to the known connection [5]. Formula (1) does not reduce to the usual formula for an arbitrary ϕ because $C_{\mathrm{so}(n)} + C_{\mathrm{so}(m)}$ does not commute with p_i and π_α although it commutes with l_i and v_α.

It is important to note that (1) can be obtained by a unitary transformation of the usual Euclidean connection, but this unitary transformation in the $-ke(n)$ representation (appropriate to describe the incoming waves) is different from the unitary transformation in the $+ke(n)$ representation (appropriate to describe the outgoing waves) where $k > 0$. If $\phi(k) = \phi(-k)$ then the two unitary transformations are the same and, of course, the S-matrix will be identical to the usual matrix, i.e. a ratio of two Γ functions.

If we take the Hamiltonian as a function of the Casimir operator of the so(n, m) algebra we can write

$$H = f\left(-\left(\frac{n+m-2}{2}\right)^2 - C_{\mathrm{so}(n,m)}\right). \tag{2}$$

Therefore, the scattering of the energy $E = k^2$, where k is the momentum, is described by representation (1) of the so(n, m) Lie algebra where $k^2 = f(\eta^2)$. We consider that f is an invertible function and, thus, we can obtain $\eta^2 = \eta^2(k^2)$. This algebraic argument fixes η^2 but the appropriate signs for the $+k$ and $-k$ representations are still undetermined. In addition to this undetermination which is pointed out for the usual Euclidean connection in [6], in our general case we have an arbitrary function $\phi(k, C_{\mathrm{so}(n,m)} + C_{\mathrm{so}(m)})$ which cannot be fixed by algebraic arguments.

Following the algebraic approach of [1], one can derive a recurrence relation for the S-matrix. We shall study the so$(n, 1)$ case, the general so(n, m) case being similar.

2.2. *The so(n, 1) algebraic scattering*

The so$(n, 1)$ Lie algebra is known as a symmetry algebra for Coulomb scattering in n dimensions. The case $n = 2$ is also known as a potential algebra for the scattering in the Pöschl–Teller or Morse potentials. In the case considered $(m = 1)$, our Euclidean connection (1) reduces to the expression of the so$(n, 1)$ operators in terms of the $e(n)$ operators:

$$
\begin{aligned}
L_i &= l_i \\
A_i &= \frac{1}{k}\mathrm{e}^{\mathrm{i}\phi(k,C_{\mathrm{so}(n)})}\left\{\frac{1}{2\mathrm{i}}[C_{\mathrm{so}(n)},\, p_i]+\eta(k)p_i\right\}\mathrm{e}^{-\mathrm{i}\phi(k,C_{\mathrm{so}(n)})}.
\end{aligned}
\tag{3}
$$

With the above Euclidean connection, the recurrence relation for the S-matrix can be written as

$$S_{l+1}(k) = \frac{e^{i\phi_+(k,l)}}{e^{i\phi_-(k,l)}} \frac{l + (n-1)/2 + i\eta_+(k)}{l + (n-1)/2 + i\eta_-(k)} S_l(k) \tag{4}$$

where

$$\begin{aligned} \Phi_+(k, l) &= \phi(k, (l+1)(l+n-1)) - \phi(k, l(l+n-2)) \\ \Phi_-(k, l) &= \phi(-k, (l+1)(l+n-1)) - \phi(-k, l(l+n-2)) \end{aligned} \tag{5}$$

and $\eta_\pm^2(k) = f^{-1}(k^2)$ if $H = f(-(\frac{n-1}{2})^2 - C_{so(n,1)})$. The usual recurrence relation [5] is obtained by taking $\Phi_+(k, l) = \Phi_-(k, l)$. We cannot determine Φ_+ and Φ_- by algebraic arguments. Therefore, a large class of S-matrices can be written in the form (4) with suitable Φ_+, Φ_-. This means that a large class of scattering problems can be treated in an algebraic framework modifying the usual Euclidean connection. Unfortunately, there is no algebraic method to obtain the appropriate forms for the Euclidean connection which correspond to a particular problem.

To justify the usefulness of the general Euclidean connection given above, we construct a Hamiltonian which is related to the so(2, 1) algebra but with a Euclidean connection different from the usual one. The operators

$$\begin{aligned} L_0 &= -i\frac{\partial}{\partial\varphi} \\ L_\pm &= e^{\pm i\varphi}\{\pm\partial_\rho + f(\rho) + c(\rho)(L_0 \pm \tfrac{1}{2})\} \end{aligned} \tag{6}$$

with

$$\begin{aligned} c(\rho) &= -\frac{(\frac{1}{2}+\alpha)\cosh\rho + (\frac{1}{2}-\alpha)\sinh\rho}{(\frac{1}{2}+\alpha)\sinh\rho + (\frac{1}{2}-\alpha)\cosh\rho} \\ f(\rho) &= \frac{\beta}{(\frac{1}{2}+\alpha)\sinh\rho + (\frac{1}{2}-\alpha)\cos\rho} \end{aligned} \tag{7}$$

and $\alpha, \beta \in \mathbb{R}$ are known to satisfy the so(1, 2) commutation relations. For $H = -\frac{1}{4} - C_{so(n,1)}$ we have $H = -\partial_\rho^2 + V_m(\rho)$ where

$$V_m(\rho) = -\frac{\beta^2 - 2m\beta[(\frac{1}{2}+\alpha)\sinh\rho + (\frac{1}{2}-\alpha)\cosh\rho] + 2\alpha(m^2 - \frac{1}{4})}{[(\frac{1}{2}+\alpha)\sinh\rho + (\frac{1}{2}-\alpha)\cosh\rho]^2}. \tag{8}$$

In this case we have the usual connection formula which yields the recurrence relation [1]

$$S_{m+1} = \frac{m + \frac{1}{2} + ik}{m + \frac{1}{2} - ik} S_m(k). \tag{9}$$

We can consider the operators

$$\tilde{L}_i = \frac{\alpha(L_0) + \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)} L_i \frac{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha(L_0) + \beta(\rho, L_0)\partial\beta(\rho, L_0)} \tag{10}$$

with $\alpha(L_0)$ a power series in L_0 with complex coefficients, and β a real valued function. Operators (10) obviously satisfy the so(1, 2) commutation relations but

$$H = -\tfrac{1}{4} - C_{\mathrm{so}(2,1)}$$
$$= \frac{\alpha(L_0) + \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)}(-\partial^2 + V_m(\rho))\frac{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha(L_0) + \beta(\rho, L_0)\partial\beta(\rho, L_0)}.$$

Operators (10) act on the linear space $\Omega = \omega \otimes h$ where ω is the complex linear space of smooth functions on $\mathbb{R}_+$ which take the zero value in the origin, and h is the Hilbert space with the orthonormal basis $\{|m\rangle, m = 0, \pm 1, \pm 2, \ldots\}$. If $\beta(0, L_0) = 0$ then the operator $\alpha(L_0)+\beta(\rho, L_0)\partial\beta(\rho, L_0)$ transform Ω into Ω and, due to the theorem of existence and unicity for the solution of a first-order differential equation, the above operator has a well defined inverse. Therefore, operators (10) are well defined on Ω. To evaluate the commutator of the operators $\frac{1}{A}$ and B we use the obvious relation $[\frac{1}{A}, B] = -\frac{1}{A}\cdot[A, B]\cdot\frac{1}{A}$. There is no ordering problem in equation (10) as the numerators and denominators commute.

If $\lim_{\rho\to\infty}\beta^2(\rho, L_0) = \epsilon(L_0)$ for the operators (10) and taking into account their asymptotic behaviour we have the recurrence relation

$$S_{m+1}(k) = \frac{\alpha(m+1) + \mathrm{i}k\epsilon(m+1)}{\alpha^*(m+1) - \mathrm{i}k\epsilon(m+1)}\frac{\alpha^*(m+1) + \mathrm{i}k\epsilon(m+1)}{\alpha(m+1) - \mathrm{i}k\epsilon(m+1)}\frac{m + \frac{1}{2} + \mathrm{i}k}{m + \frac{1}{2} - \mathrm{i}k}$$
$$\times \frac{\alpha^*(m) - \mathrm{i}k\epsilon(m)}{\alpha(m) + \mathrm{i}k\epsilon(m)}\frac{\alpha(m) - \mathrm{i}k\epsilon(m)}{\alpha^*(m) + \mathrm{i}k\epsilon(m)}S_m(k) \tag{11}$$

which can be solved to yield

$$S_m(k) = \mathrm{e}^{\mathrm{i}\xi(k)}\frac{\Gamma(m + \frac{1}{2} + \mathrm{i}k)}{\Gamma(m + \frac{1}{2} - \mathrm{i}k)}\frac{(\mathrm{Im}\,\alpha(m))^2 - (\epsilon(m)k + \mathrm{i}\,\mathrm{Re}\,\alpha(m))^2}{(\mathrm{Im}\,\alpha(m))^2 - (\epsilon(m)k - \mathrm{i}\,\mathrm{Re}\,\alpha(m))^2} \tag{12}$$

which is not a ratio of Γ functions. In the particular case $\mathrm{Im}\,\alpha(m) = m$, $\mathrm{Re}\,\alpha(m) = a$, $\epsilon(m) \equiv \epsilon$ and denoting $\gamma(k) = \epsilon k + \mathrm{i}a$ we have

$$S_m(k) = \frac{\Gamma(m + \frac{1}{2} + \mathrm{i}k)}{\Gamma(m + \frac{1}{2} - \mathrm{i}k)}\frac{m^2 - \gamma^2(k)}{m^2 - \gamma^{2*}(k)}. \tag{13}$$

The above form of the S-matrix, i.e. a rational function in angular momentum m, was introduced earlier on heuristic grounds by Remler [7] for atomic collisions, and used in heavy-ion scattering [8]. The WKB inversion for this simple analytic form of the S-matrix can be carried out analytically [8]. The quantal inversion uses the logarithmic derivatives of the Jost solutions in the reference potential [8] (these logarithmic derivatives are the solutions of a Riccati equation with appropriate boundary conditions). When we use transformation (10), $\Phi_m = \frac{\alpha+\beta\partial\beta}{\alpha^*-\beta\partial\beta}\Psi_m$ where $\{\Psi_m\}$ is the standard basis for the realization L_i, and $\{\Phi_m\}$ is the standard basis for the realization $\tilde{L}_i$. Thus, we have

$$\{\alpha^*(m) - \beta(\rho, m)\partial\beta(\rho, m)\}\Phi_m(\rho) = \{\alpha(m) + \beta(\rho, m)\partial\beta(\rho, m)\}\Psi_m(\rho).$$

The boundary condition at the origin $\Psi_m(0) = 0$ is preserved if $\beta(0, m) = 0$ and $\alpha(m) \neq 0$. These conditions ensure that operators (10) are well defined.

Therefore, to preserve the boundary condition at the origin we have to use (10) with $\beta(0, m) = 0$ (e.g. $\beta(\rho) = \tanh\rho$); the case $\beta(\rho, m) \equiv 0$ is trivial. Transformation (10) can be used iteratively to obtain examples of Hamiltonians with different S-matrix recurrence relations. The new recurrence relations are the usual relations provided that $\lim_{\rho\to\infty}\beta(\rho, L_0) = 0$. The Hamiltonian connected to the realization (10) can be put in the form

$$H = \frac{\alpha(L_0) + \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)} H \frac{\alpha^*(L_0) - \beta(\rho, L_0)\partial\beta(\rho, L_0)}{\alpha(L_0 + \beta(\rho, L_0)\partial\beta(\rho, L_0)} \tag{14}$$

where H is the Hamiltonian connected to the realization given by the L_i operators.

3. Conclusions

We obtain a general Euclidean connection (1) which contains the usual connection as a particular case. This Euclidean connection contains an arbitrary function which cannot be fixed by algebraic means. A large class of scattering problems can now be treated in an algebraic framework by modifying the usual Euclidean connection but, unfortunately, there is no algebraic method available to choose the appropriate Euclidean connection. Also, we construct a Hamiltonian (14), for which the appropriate Euclidean connection is not the usual connection. The algebraic S-matrix for this Hamiltonian could be a rational function in angular momentum (up to a factor). Such rational representations of the scattering function were used in atom–atom scattering and heavy-ion collisions.

The present approach allows us to obtain many examples of Hamiltonians for which the usual S-matrix recurrence relation holds by starting with a known example and using the unitary transformation (10) with $\lim_{\rho\to\infty}\beta(\rho) = 0$. It is important to note that the case β = constant is not allowed and that all the obtained Hamiltonians contain momentum dependent potentials.

References

[1] Alhassid Y, Gursey F and Iachello F 1983 *Ann. Phys., NY* **148** 346; 1986 *Ann. Phys., NY* **167** 181
Wu J, Iachello F and Alhassid Y 1987 *Ann. Phys., NY* **173** 68
Frank A and Wolf K B 1984 *Phys. Rev. Lett.* **52** 37; 1985 *J. Math. Phys.* **28** 973

[2] Alhassid Y, Iachello F and Shao B 1988 *Phys. Lett.* **201B** 183
Amos K, Berge L, Allen L J and Fiedeldey H 1990 *Phys. Rev. Lett.* **64** 625; 1991 *Phys. Lett.* **269B** 19; 1993 *Phys. Rev.* C **47** 2827

[3] Alhassid Y and Iachello F 1989 *Nucl. Phys.* A **501** 585
Lepine-Szily A, Obuti M M, Lichtenthaler Filho R, Oliveira J M and Villari A C C 1990 *Phys. Lett.* **243B** 23
Lichtenthaler R, Villari A C C, Gomes L C and Carrilho Soares P 1991 *Phys. Lett.* **269B** 49

[4] Wu J 1987 *J. Phys. A: Math. Gen.* **20** 4637; 1985 *PhD Thesis* Yale University

[5] Frank A, Alhassid Y and Iachello F 1986 *Phys. Rev.* A **34** 677

[6] Zielke A 1990 *Phys. Rev.* A **42** 1358
Zielke A and Scheid W 1993 *J. Phys. A: Math. Gen.* **26** 2047

[7] Remler E A 1971 *Phys. Rev.* A **3** 1949

[8] Fiedeldey H, Lipperheide R, Naidoo K and Sofianos S A 1984 *Phys. Rev.* C **30** 434
Lipperheide R and Fiedeldey H 1981 *Z. Phys.* A **301** 81
Naidoo K, Fiedeldey H, Sofianos S A and Lipperheide R 1984 *Nucl. Phys.* A **149** 13